Functional Keldysh theory of spin torques

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We present a microscopic treatment of current-induced torques and thermal fluctuations in itinerant ferromagnets based on a functional formulation of the Keldysh formalism. We find that the nonequilibrium magnetization dynamics is governed by a stochastic Landau-Lifschitz-Gilbert equation with spin-transfer torques. We calculate the Gilbert damping parameter α and the nonadiabatic spin transfer torque parameter β for a model ferromagnet. We find that $\beta \neq \alpha$, in agreement with the results obtained using imaginary-time methods of Kohno *et al.* [J. Phys. Soc. Jpn. **75**, 113706 (2006)]. We comment on the relationship between *s-d* and isotropic-Stoner toy models of ferromagnetism and more realistic density-functional-theory models, and on the implications of these relationships for predictions of the β/α ratio which plays a central role in domain-wall motion. Only for a single-parabolic-band isotropic-Stoner model with an exchange splitting that is small compared to the Fermi energy does β/α approach 1. In addition, our microscopic formalism naturally incorporates the fluctuations needed in a nonzero-temperature description of the magnetization. We find that to first order in the applied electric field, the usual form of thermal fluctuations via a phenomenological stochastic magnetic field holds.

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I. INTRODUCTION

Phenomena related to order-parameter manipulation by transport currents have recently received a great deal of attention in magnetic metals and magnetic semiconductors. Spin-transfer torques, which lead to current-driven nanomagnet reversal and to domain-wall motion in narrow wires, have been at the center of this activity.[1–28] In spin-transfer-torque theory, the usual Landau-Lifschitz-Gilbert (LLG) equation of motion for the magnetization direction $\hat{\Omega}$ acquires terms corresponding to the so-called adiabatic and nonadiabatic spin-transfer torques which are both proportional to current. Both torques can be constructed from symmetry arguments by requiring that they be orthogonal to the magnetization direction and by realizing that the current essentially breaks inversion symmetry. The latter implies that, in the long-wavelength limit, terms proportional to $\nabla\hat{\Omega}$ are allowed in the LLG equation of motion. The adiabatic spin-transfer torque[3,4] is defined as $-(\mathbf{v}_s\cdot\nabla)\hat{\Omega}$, where $\mathbf{v}_s$ is a velocity, proportional to the current, that characterizes the efficiency of spin transfer and is required for dimensional reasons. The nonadiabatic spin-transfer torque[10] is given by $-\beta\hat{\Omega}\times(\mathbf{v}_s\cdot\nabla)\hat{\Omega}$ and is characterized by the dimensionless parameter β.

The LLG equation that incorporates both spin transfer torques is then given by

$$\left(\frac{\partial}{\partial t}+\mathbf{v}_s\cdot\nabla\right)\hat{\Omega}-\hat{\Omega}\times\mathbf{H}=-\alpha\hat{\Omega}\times\left(\frac{\partial}{\partial t}+\frac{\beta}{\alpha}\mathbf{v}_s\cdot\nabla\right)\hat{\Omega}, \tag{1}$$

in the long-wavelength low-frequency limit,[29] where $\mathbf{H}$ is the effective field and α the Gilbert damping constant. The coefficients α and β are dissipative in the sense that they relate quantities that are even under time reversal to quantities that are odd under time reversal.[30,31] The observation that both α and β are related to dissipation is important in the context of nonzero-temperature effects which play an especially important role[20] in experiments with magnetic semiconductors.[27,28] Nonzero temperature is usually accounted for by adding a Gaussian stochastic magnetic field $\mathbf{h}$ to the effective field in the LLG equation.[32–39] The strength of this noisy magnetic field is related to the Gilbert damping by the fluctuation-dissipation theorem, which ensures that the system is characterized in equilibrium by the appropriate Boltzmann distribution. *A priori*, the generalization of Eq. (1) to nonzero temperatures is not clear, since both α and β correspond to dissipative processes and the fluctuation-dissipation theorem need not be valid because the nonzero current implies that the system is out of equilibrium.

As noted in the literature on current-driven domain-wall motion,[12,17] the case of $\beta=\alpha$ is special because both sides of Eq. (1) then contain the "comoving" derivative $D/Dt=\partial/\partial t+\mathbf{v}_s\cdot\nabla$, so that the equation of motion admits solutions $\hat{\Omega}_d(t)=\hat{\Omega}_0(\mathbf{x}-\mathbf{v}_s t)$, where $\hat{\Omega}_0(\mathbf{x})$ is a time-independent solution of the LLG equation in the absence of currents. The solution $\hat{\Omega}_d(t)$ corresponds to "drift" of static magnetization textures with velocity $\mathbf{v}_s$. Arguing that these solutions must exist, Barnes and Maekawa[12] claim that $\beta=\alpha$. However, in realistic systems, there is no Galilean invariance[4] that requires the existence of such solutions and therefore, in general, $\beta\neq\alpha$.[18,19] Instead, the nonadiabatic spin-transfer torque acquires contributions from all microscopic processes that violate spin conservation and therefore correspond to terms in the microscopic Hamiltonian that are not invariant under spin rotations. Such processes also contribute to the Gilbert damping term, and therefore, in principle, any nonzero Gilbert damping parameter α implies nonzero nonadiabatic spin-transfer torques, as we show in our specific microscopic model calculations.

The LLG equation [Eq. (1)] is motivated mainly by symmetry considerations and contains four different quantities whose meaning can be specified precisely only by a microscopic theory which details, at least in principle, precisely how they should be evaluated given the full system Hamiltonian. These quantities are (i) the effective magnetic field $\mathbf{H}$, (ii) the transport spin velocity $\mathbf{v}_s$, (iii) the Gilbert damping parameter α, and (iv) the nonadiabatic spin-transfer torque parameter β. The effective magnetic field $\mathbf{H}$ includes the external magnetic field and additional contributions due to magnetostatic interactions and magnetocrystalline anisotropy. The physics of $\mathbf{H}$ is well understood[40] and not the subject of this paper. The three remaining quantities emerge in a microscopic theory from the slow (up to first order in time derivatives or frequency ω) smooth (up to first order in space derivatives or wave vector $\mathbf{q}$) response of the magnetization direction to an external magnetic field, in the presence of an external electric field which drives a transport current. The coefficient α then emerges as the ratio of the reactive and dissipative contributions that appear at first order in ω in this response function. When spin-orbit interactions are neglected, it is easy to verify that the coefficient of the reactive term in the total spin response is the unperturbed spin density, explaining the unit value of this coefficient in Eq. (1). The two first-order space derivative terms in this equation reflect, respectively, the change in the reactive and dissipative responses due to an external electric field. Like its zero-current counterpart, the current-related reactive terms can be understood in quite general terms based only on spin-conservation considerations, while the dissipative term is sensitive to microscopic details.

As explained above, the condition $\beta = \alpha$ corresponds to Galilean invariance at a macroscopic level. Since the dissipative terms emerge from spin-dependent disorder (or spin-independent disorder when spin-orbit interactions are included in the crystal band structure), it is clear that Galilean invariance does not hold microscopically. Our calculations show that $\beta \approx \alpha$ can occur in models with very specific properties,[17] but does not occur in general. For example, $\beta \approx \alpha$ occurs in the specific toy model that we study below only when the ferromagnetism is weak in the sense that the exchange splitting is much smaller than the Fermi energy.[17] We believe that we obtain this result only because the model has isotropic parabolic bands and that $\beta = \alpha$ (macroscopic Galilean invariance) can occur only accidentally in systems with either realistic bands or realistic disorder. In the important transition-metal ferromagnet spintronic materials, in particular, we will argue that orbitals which have dominant d character contribute more strongly to the magnetization than to transport and that β will tend to be larger than α as a consequence.

In this paper, we present a microscopic derivation of the equation of motion in Eq. (1) of the direction of magnetization in the presence of current *and* at finite temperature. We use a functional formulation of the Keldysh nonequilibrium formalism[41,42] which leads, in a natural way, to the path-integral formulation of stochastic differential equations.[43,44] Within our microscopic treatment, the dissipative nature of α and β is explicit because, as briefly discussed above and to be shown in more detail, they follow from the dissipative part of the spin-density–spin-density response function and photon–two-magnon interaction vertex, respectively. We focus on the simple microscopic toy model used in previous work[17,18] that is intended to provide a qualitative description of a generic ferromagnet and includes disorder and short-range repulsive electron-electron interactions. The model's ferromagnetism is treated at the level of the Stoner mean-field theory. For the disorder, we use the same model as in Ref. 18 and, where applicable, our results for α and β agree with theirs. Our random-phase-approximation treatment of Stoner quasiparticle fluctuations evinces the equivalence of Stoner and s-d models, in the sense that in both models the quantities α and β are determined by the same response function. In particular, $\alpha \neq \beta$ for both models in general.

One benefit of the concinnity of the functional formulation of the Keldysh formalism is that it enables a natural determination of the thermal fluctuations without explicitly appealing to the fluctuation-dissipation theorem. We find that to lowest order in the applied electric field, the form usually assumed for the strength of the fluctuations holds and that there is no contribution to the white-noise thermal fluctuations that is related to the nonadiabatic torque. We emphasize that this formalism, which has been reviewed in other publications and applied to other problems,[41,42] is similar in structure to the functional formulation of standard equilibrium Green's functions for linear-response theory but is more powerful for nonequilibrium and nonlinear problems.

Since the formalism we use may not be familiar to most readers, we first present the model and main results in a separate section, namely, Sec. II. In Sec. III, we present the formalism and outline the calculations. In the Appendix, we carry out a typical calculation in more detail. Both Sec. III and the Appendix may be skipped by readers who are familiar with the formalism or who may be more interested in the results obtained. We end in Sec. IV with our conclusions.

II. MODEL AND SUMMARY OF RESULTS

We model the disordered itinerant ferromagnet as electrons with delta-function-like repulsive interactions using the Hamiltonian

$$H[\hat{\boldsymbol{\psi}}^\dagger,\hat{\boldsymbol{\psi}}] = \int d\mathbf{x}\Bigg\{\hat{\boldsymbol{\psi}}^\dagger(\mathbf{x},t)\Bigg[-\frac{\hbar^2\nabla^2}{2m} - \frac{\Delta_{\rm ext}}{2}\tau_z + V_0(\mathbf{x}) + V_a(\mathbf{x})\tau_a\Bigg]\hat{\boldsymbol{\psi}}(\mathbf{x},t) + \frac{1}{c}\hat{\mathbf{J}}(\mathbf{x},t)\cdot\mathbf{A}(\mathbf{x},t) + U\hat{\psi}^\dagger_\uparrow(\mathbf{x},t)\hat{\psi}^\dagger_\downarrow(\mathbf{x},t)\hat{\psi}_\downarrow(\mathbf{x},t)\hat{\psi}_\uparrow(\mathbf{x},t)\Bigg\}, \tag{2}$$

where for notational convenience we have introduced the spinor

$$\hat{\boldsymbol{\psi}}(\mathbf{x},t) = \begin{pmatrix}\hat{\psi}_\uparrow(\mathbf{x},t)\\ \hat{\psi}_\downarrow(\mathbf{x},t)\end{pmatrix}. \tag{3}$$

In these expressions, the Heisenberg operators $\hat{\psi}_\sigma(\mathbf{x},t)$ annihilate an electron in the spin state labeled by $\sigma \in \{\uparrow,\downarrow\}$ and obey the usual equal-time commutation relations. These spin states have their quantization axis parallel to an external Zeeman magnetic field in the z direction, which contributes Δ_{ext} to the energy difference between minority and majority spins. Note that in Eq. (2), the Pauli matrices are indicated by τ_a and that a sum over the repeated index $a \in \{x,y,z\}$ is implied. The free-electron dispersion at momentum $\hbar\mathbf{k}$, given by $\epsilon_{\mathbf{k}}=\hbar^2\mathbf{k}^2/2m$, is parabolic with an effective mass m ($\hbar$ denotes Planck's constant).

We choose a delta-function interaction with strength U because then the field-theoretic procedure to introduce the magnetization direction as a dynamic variable is easier to implement. This so-called Hubbard-Stratonovich transformation[45] can also be generalized to spatially nonlocal interactions.[46] This procedure, to be discussed in more detail in the next section, also yields the mean-field, i.e., Stoner, saddle-point equation for the exchange-interaction contribution to the spin splitting

$$\Delta = U \int \frac{d\mathbf{k}}{(2\pi)^3}\left\{ N_F\left[\epsilon_{\mathbf{k}} - \frac{(\Delta+\Delta_{\text{ext}})}{2} - \mu\right] - N_F\left[\epsilon_{\mathbf{k}} + \frac{(\Delta+\Delta_{\text{ext}})}{2} - \mu\right]\right\} = U\rho_s, \tag{4}$$

where $N_F(x)=[\exp(x/k_BT)+1]^{-1}$ is the Fermi distribution function with k_BT the thermal energy, μ is the chemical potential that includes a Hartree mean-field shift, and ρ_s is the magnetization density. In practice, we do not explicitly determine the exchange splitting from this equation but simply assume that Δ is a solution whose value may be determined from experiment if needed. This is another reason for simply using a delta-function interaction.

For the disorder, we use the same model as Kohno *et al.*[18] in which the spin-dependent disorder potentials are characterized by

$$\overline{V_a(\mathbf{x})V_b(\mathbf{x}')} = \sigma_a \delta(\mathbf{x}-\mathbf{x}')\delta_{ab}, \tag{5}$$

where $\overline{\cdots}$ indicates averaging over different realizations of the disorder. For randomly distributed scatterers,

$$\sigma_{x,y} = n_s u_s^2 \overline{S_\perp^2}, \quad \sigma_z = n_s u_s^2 \overline{S_z^2}, \quad \sigma_0 = n_i u_i^2, \tag{6}$$

where $u_i(u_s)$ and $n_i(n_s)$ are the strength and density of the scatterer charge (spin) component, respectively, and $\overline{S_a^2}$ denotes the average scatterer field orientation. Within the self-consistent Born approximation, the decay rate γ_σ and lifetime τ_σ^{sc} of a plane wave with spin state $|\sigma\rangle$ are determined from

$$\hbar\gamma_\sigma \equiv \frac{\hbar}{2\tau_\sigma^{\text{sc}}} = \pi n_i u_i^2 \nu_\sigma + \pi n_s u_s^2 (2\overline{S_\perp^2}\nu_{-\sigma} + \overline{S_z^2}\nu_\sigma), \tag{7}$$

where the density of states per spin at the Fermi level $\nu_\sigma = mk_{F\sigma}/2\pi^2\hbar^2$ and the Fermi wave number $k_{F\sigma} = \sqrt{2m(\epsilon_F+\sigma M)/\hbar^2}$, where $M=(\Delta+\Delta_{\text{ext}})/2$ is the total spin splitting.

Finally, the current in our theory is induced by an external homogeneous electric field $\mathbf{E}$ that, in the London gauge, is related to the vector potential by

$$\mathbf{A}(t) = \frac{c\mathbf{E}}{i\omega_p} e^{-i\omega_p t}, \tag{8}$$

where ω_p is the frequency of the electric field, to be taken to zero eventually, and c is the speed of light. In the Hamiltonian [Eq. (2)], the vector potential is minimally coupled to the electrons via the charge current-density operator

$$\hat{\mathbf{J}}(\mathbf{x},t) = \frac{i|e|\hbar}{2m}[\hat{\boldsymbol{\psi}}^\dagger(\mathbf{x},t)\nabla\hat{\boldsymbol{\psi}}(\mathbf{x},t) - (\nabla\hat{\boldsymbol{\psi}}^\dagger(\mathbf{x},t))\hat{\boldsymbol{\psi}}(\mathbf{x},t)], \tag{9}$$

with $-|e|$ the electron charge. In the above expression we have omitted the diamagnetic contribution as it plays no role in the following.

In the next section, we derive, starting from the Hamiltonian in Eq. (2), the equations of motion for long-wavelength deviations $\delta\boldsymbol{\Omega}$ of the magnetization direction from the collinear ground state, defined by $\hat{\Omega}=\hat{z}+\delta\boldsymbol{\Omega}$. We find that these transverse deviations obey the following stochastic equations of motion:

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_s\cdot\nabla\right)\delta\Omega_a = \epsilon_{ab}\left[\frac{\Delta_{\text{ext}}}{\hbar}\delta\Omega_b + \alpha\left(\frac{\partial}{\partial t} + \frac{\beta}{\alpha}\mathbf{v}_s\cdot\nabla\right)\delta\Omega_b - h_b\right], \tag{10}$$

where a sum over repeated transverse indices $a,b\in\{x,y\}$ is implied and ϵ_{ab} is the two-dimensional Levi-Civita tensor. The Gilbert damping parameter is given by

$$\alpha = \frac{2\pi}{\rho_s}\{n_s u_s^2[\overline{S_\perp^2}(\nu_\uparrow^2+\nu_\downarrow^2) + 2\overline{S_z^2}\nu_\uparrow\nu_\downarrow]\}, \tag{11}$$

with the magnetization density $\rho_s=\Delta/U$. (For the s-d model, ρ_s corresponds to the carrier-spin-polarization density.) The velocity $\mathbf{v}_s$ is related to the electric field by

$$\mathbf{v}_s = \frac{|e|\mathbf{E}}{m\rho_s}(n_\uparrow\tau_\uparrow^{\text{sc}} - n_\downarrow\tau_\downarrow^{\text{sc}}), \tag{12}$$

in terms of the density of majority and minority electrons, denoted by $n_\uparrow$ and $n_\downarrow$, respectively. Using the fact that to linear order in the electric field the current densities of the majority and minority electrons are determined from $\mathbf{j}_\sigma = n_\sigma|e|^2\tau_\sigma^{\text{sc}}\mathbf{E}/m$, we observe that the expression for $\mathbf{v}_s$ reduces to the usual expression $\mathbf{v}_s=(\mathbf{j}_\uparrow-\mathbf{j}_\downarrow)/(-|e|\rho_s)$. Our result for the β parameter reads

$$\beta = \frac{2\pi n_s u_s^2}{M}\left[\frac{n_\uparrow\tau_\uparrow^{\text{sc}}(\overline{S_z^2}\nu_\downarrow + \overline{S_\perp^2}\nu_\uparrow) - n_\downarrow\tau_\downarrow^{\text{sc}}(\overline{S_z^2}\nu_\uparrow + \overline{S_\perp^2}\nu_\downarrow)}{(n_\uparrow\tau_\uparrow^{\text{sc}} - n_\downarrow\tau_\downarrow^{\text{sc}})}\right]. \tag{13}$$

Notice that, as expected, only spin-dependent scattering contributes to the nonadiabatic torque parameter β and the Gilbert damping parameter α.

In addition, we find that the thermal fluctuations via the stochastic magnetic field $\mathbf{h}$ are determined by

$$\langle h_a(\mathbf{x},t)h_b(\mathbf{x}',t')\rangle_{\text{noise}} = \frac{2\alpha k_B T}{\hbar(\rho_s/2)}\delta(\mathbf{x}-\mathbf{x}')\delta(t-t')\delta_{ab}, \tag{14}$$

where the average is over different realizations of the noise. We stress that this form for the strength of the fluctuations is derived explicitly, without appealing to the fluctuation-dissipation theorem. This is important since it is not *a priori* obvious that in the current-carrying situation the fluctuation-dissipation theorem holds. The form of the strength of the fluctuations in Eq. (14) is, however, of the usual form, i.e., it is of the same form as inferred by the equilibrium fluctuation-dissipation theorem. This result comes about because shot-noise[47] contributions to the magnetization noise[48,49] enter as higher-order terms in the applied electric field than the linear response in electric field considered here.

The linear-response result in Eq. (10) is consistent up to $\mathcal{O}(\delta\mathbf{\Omega})$ with the Landau-Lifschitz-Gilbert equation that includes both nonadiabatic and adiabatic spin-transfer torques and thermal fluctuations,

$$\left(\frac{\partial}{\partial t}+\mathbf{v}_s\cdot\nabla\right)\hat{\Omega}-\hat{\Omega}\times(\mathbf{H}+\mathbf{h}) = -\alpha\hat{\Omega}\times\left(\frac{\partial}{\partial t}+\frac{\beta}{\alpha}\mathbf{v}_s\cdot\nabla\right)\hat{\Omega}, \tag{15}$$

where $\mathbf{h}$ is a stochastic magnetic field that obeys the correlations given by Eq. (14).

We end this section by sketching how the various results come about. In the theory to be discussed in more detail in the next section, the two quantities of interest are the transverse spin-density–spin-density response function (or magnon propagator), which determines the Gilbert damping parameter, and the photon–two-magnon interaction vertex which gives rise to both the adiabatic and nonadiabatic spin-transfer torques. (Note that the photons simply correspond to the external electric field in this case.) Feynman diagrams for both of these functions are given in Figs. 1(a) and 1(b). Quite generally, the response function in Fig. 1 has reactive and dissipative parts. In the long-time and length scale expansion corresponding to the LLG equations, the small-frequency zero-momentum part of the reactive contribution gives rise to the time derivative on the left-hand side of Eq. (10). (Note that after Fourier transformation, frequencies turn into time derivatives.) The small-frequency zero-momentum part of the dissipative contribution to the same response function determines the Gilbert damping term on the right-hand side of Eq. (10). Physically, this dissipative contribution comes from spin waves that decay into particle-hole excitations. Energy conservation then leads a delta-function-like, i.e., dissipative, contribution of the form $\delta(\hbar\omega-\epsilon_1+\epsilon_2)$, where $\hbar\omega$ is the energy of the spin wave and $\epsilon_1-\epsilon_2$ the energy of the particle-hole pair. Summing over all possible particle-hole pair energies and performing a zero-momentum low-frequency expansion then lead to the Gilbert damping term on the right-hand side of Eq. (10). Similarly, the spatial derivatives on the left-hand side of Eq. (10) are the result of the reactive contribution to the zero-frequency small-momentum behavior of the photon spin-wave interaction vertex and give rise to the adiabatic spin-transfer torque. The nonadiabatic

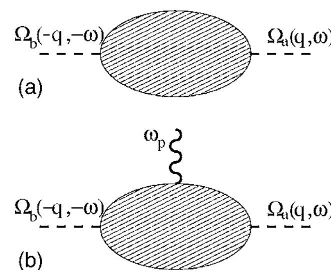

FIG. 1. (a) Feynman diagram for the transverse spin-density–spin-density response function. This diagram may equivalently be thought of as the spin-wave, or magnon, propagator. The magnon frequency is denoted by ω and its momentum by $\mathbf{q}$. (b) Photon–two-magnon interaction vertex that ultimately gives rises to spin-transfer torques. (Note that the photons correspond to the external electric field used in our theory to induce transport current and hence to terms proportional to the current within linear response.) This vertex describes the interaction of spin waves with frequency ω and momentum $\mathbf{q}$ with the electric current that is generated by an external electric (photon) field of frequency ω_p. (The theory of spin-transfer torques is related to the $\omega_p\to 0$ limit of this diagram and this limit is taken in the Feynman diagram.)

torque, proportional to β on the right-hand side of Eq. (10), then emerges from the dissipative part of the interaction vertex and gets contributions from physical processes in which a spin wave interacts with the current and subsequently decays into an incoherent particle-hole excitation.

III. NONEQUILIBRIUM MAGNETIZATION DYNAMICS

In this section, we derive the stochastic equation of motion for the transverse magnetization in the presence of current. We start out by deriving the general equations and subsequently give the results for the long-wavelength low-frequency limit. We discuss the equilibrium situation, i.e., the case without electric field, and the nonequilibrium situation separately.

A. Stochastic equations of motion

Our starting point is the path-integral expression for the coherent-state probability distribution, written as a functional integral[41]

$$P[\boldsymbol{\phi}^*,\boldsymbol{\phi};t] = \int d[\psi_\uparrow^*]d[\psi_\uparrow]d[\psi_\downarrow^*]d[\psi_\downarrow]\exp\left\{\frac{i}{\hbar}S[\boldsymbol{\psi}^*,\boldsymbol{\psi}]\right\}. \tag{16}$$

Roughly speaking, this distribution specifies the probability for the system to be in the Grassman coherent state $\boldsymbol{\phi}(\mathbf{x},t)$. The action is expressed in terms of the fermionic fields $\boldsymbol{\psi}$ and $\boldsymbol{\psi}^*$ by

$$S[\boldsymbol{\psi}^*,\boldsymbol{\psi}] = \int_{\mathcal{C}^t} dt' \int d\mathbf{x}\left\{\boldsymbol{\psi}_\sigma^*(\mathbf{x},t')i\hbar\frac{\partial\boldsymbol{\psi}_\sigma(\mathbf{x},t')}{\partial t'} - H[\boldsymbol{\psi}^*(\mathbf{x},t'),\boldsymbol{\psi}(\mathbf{x},t')]\right\}. \tag{17}$$

The functional integration in Eq. (16) is over all fields evolving forward in time from $-\infty$ to t, and back, thereby defining the time integration in the action in Eq. (17) to be over the Keldysh contour $\mathcal{C}^t$.

We rewrite the interaction term as[50]

$$U\psi_\uparrow^* \psi_\downarrow^* \psi_\downarrow \psi_\uparrow = \frac{U}{4}(\boldsymbol{\psi}^* \boldsymbol{\psi})^2 - \frac{U}{4}(\boldsymbol{\psi}^* \boldsymbol{\tau} \cdot \hat{n} \boldsymbol{\psi})^2, \tag{18}$$

with $\hat{n}(\mathbf{x},t)$ an arbitrary unit vector that determines the spin quantization axis. Functional integration over the latter enforces rotation invariance.[50] The interaction terms on the right-hand side of Eq. (18) are decoupled by writing them as a Gaussian functional integral over a density field $\langle \rho(\mathbf{x},t)\rangle = \langle \boldsymbol{\psi}^* \boldsymbol{\psi}\rangle$ and spin-density field $\langle \Delta(\mathbf{x},t)\hat{n}(\mathbf{x},t)\rangle = U\langle \boldsymbol{\psi}^* \boldsymbol{\tau} \boldsymbol{\psi}\rangle/2$, respectively. [The precise meaning of the angular brackets $\langle \cdots \rangle$ is defined below Eq. (23).] This Hubbard-Stratonovich transformation[41,45,50] then introduces the density and spin density as dynamical variables in the path integral in Eq. (16). Density and spin-density amplitude fluctuations are gapped and can be approximated at low temperatures and energies by their saddle-point values. For the density, we then find a Hartree-Fock equation, giving rise to a mean-field Hartree shift which we absorb in the chemical potential. For the spin-density amplitude, we find the saddle-point equation for Δ in Eq. (4).

After these steps, we ultimately find that the probability distribution is given by

$$P[\boldsymbol{\phi}^*, \boldsymbol{\phi}, \hat{\Omega}; t] = \int d[\psi_\uparrow^*] d[\psi_\uparrow] d[\psi_\downarrow^*] d[\psi_\downarrow] d[\hat{n}] \exp\left\{ \frac{i}{\hbar} S'[\boldsymbol{\psi}^*, \boldsymbol{\psi}, \hat{n}] \right\}, \tag{19}$$

where the unit vector $\hat{\Omega}$ enters as the boundary condition at $t'=t$ on the functional integration over the fluctuating magnetization orientation $\hat{n}$. We do not explicitly indicate the boundary condition on the fermion fields, because, as we shall see, the quantity that enters is the fermion Green's function which is determined without explicitly referring to the boundary conditions. The action $S'[\boldsymbol{\psi}^*, \boldsymbol{\psi}, \hat{n}]$ is, using the same notation as for the Hamiltonian in Eq. (2), explicitly given by

$$S'[\boldsymbol{\psi}^*, \boldsymbol{\psi}, \hat{n}] = \int_{\mathcal{C}^t} dt' \int d\mathbf{x} \Bigg\{ \boldsymbol{\psi}^*(\mathbf{x},t') \Bigg[i\hbar \frac{\partial}{\partial t'} + \frac{\hbar^2 \nabla^2}{2m} + \frac{\Delta}{2}\hat{n}(\mathbf{x},t') \cdot \boldsymbol{\tau} + \frac{\Delta_{\text{ext}}}{2}\tau_z - V_0(\mathbf{x}) - V_a(\mathbf{x})\tau_a \Bigg] \times \boldsymbol{\psi}(\mathbf{x},t') - \frac{1}{c}\mathbf{J}(\mathbf{x},t') \cdot \mathbf{A}(t') \Bigg\}. \tag{20}$$

At this point, we note that, if we would add a separate Berry-phase term in this action to enforce the angular-momentum-like quantization of $\hat{n}$, the resulting action would be the starting point for treating the s-d model.

We now do perturbation theory around the collinear state by writing

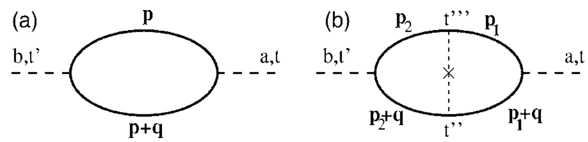

FIG. 2. (a) Lowest-order diagram and (b) first vertex correction to the spin-density–spin-density response function. The momentum of the spin wave is denoted by $\mathbf{q}$. Impurity scattering is indicated by the thin dashed line where the $\times$ indicates the position of the impurity.

$$\hat{n}(\mathbf{x},t) \simeq \begin{pmatrix} \delta n_x(\mathbf{x},t) \\ \delta n_y(\mathbf{x},t) \\ 1 - \frac{1}{2}[\delta n_x(\mathbf{x},t)]^2 - \frac{1}{2}[\delta n_y(\mathbf{x},t)]^2 \end{pmatrix} \tag{21}$$

and integrating out the electronic fields using second-order perturbation theory in $\delta n_a(\mathbf{x},t)$ and first-order perturbation theory in $\mathbf{A}(\mathbf{x},t)$. (Note that to find an equation of motion for δn_a that is valid up to first order, the action needs to be determined up to quadratic terms in δn_a.) To this order, the effective action for the magnetization is written in the form

$$S_{\text{eff}}[\delta \mathbf{n}] = \int_{\mathcal{C}^t} dt' \int d\mathbf{x} \Bigg\{ -\frac{\Delta}{4}\rho_s \delta n_a^2(\mathbf{x},t') + \int_{\mathcal{C}^t} dt'' \int d\mathbf{x}' \, \delta n_a(\mathbf{x},t') \times \Pi_{ab}(\mathbf{x}-\mathbf{x}';t',t'') \delta n_b(\mathbf{x}',t'') + \int_{\mathcal{C}^t} dt'' \int d\mathbf{x}' \, \delta n_a(\mathbf{x},t') \times K_{ab}(\mathbf{x}-\mathbf{x}';t',t'') \delta n_b(\mathbf{x}',t'') \Bigg\}, \tag{22}$$

where a sum over repeated transverse indices $a,b \in \{x,y\}$ is again implied. In Eq. (22), the function $\Pi_{ab}(\mathbf{x};t,t')$ is determined by the transverse spin-density–spin-density response function

$$\Pi_{ab}(\mathbf{x}-\mathbf{x}';t,t') = \frac{i\Delta^2}{8\hbar}\langle \boldsymbol{\psi}^\dagger(\mathbf{x},t)\tau_a \boldsymbol{\psi}(\mathbf{x},t) \boldsymbol{\psi}^\dagger(\mathbf{x}',t')\tau_b \boldsymbol{\psi}(\mathbf{x}',t')\rangle, \tag{23}$$

shown diagrammatically in Fig. 1(a). In this expression, the angular brackets $\langle \cdots \rangle \equiv \text{Tr}[\hat{\rho}(-\infty)\cdots]$ denote an average with respect to the density matrix $\hat{\rho}(-\infty)$ of a system of electrons with the action in Eq. (20), with $\hat{n}(\mathbf{x},t) = \hat{z}$ and $\mathbf{A}=0$, which is in equilibrium. The function $K_{ab}(\mathbf{x};t,t')$ is given by

$$K_{ab}(\mathbf{x}-\mathbf{x}';t,t') = \int_{\mathcal{C}^t} dt'' \int d\mathbf{x}'' \left[\frac{\Delta^2}{8\hbar^2 c} \boldsymbol{\Lambda}_{ab}(\mathbf{x},\mathbf{x}',\mathbf{x}'';t,t',t'') \cdot \mathbf{A}(t'') \right], \tag{24}$$

where the photon–two-magnon vertex function

$$\begin{aligned}\mathbf{\Lambda}_{ab}(\mathbf{x},\mathbf{x}',\mathbf{x}'';t,t',t'') \\ = \langle \boldsymbol{\psi}^{\dagger}(\mathbf{x},t')\tau_a\boldsymbol{\psi}(\mathbf{x},t')\boldsymbol{\psi}^{\dagger}(\mathbf{x}',t')\tau_b\boldsymbol{\psi}(\mathbf{x}',t')\mathbf{J}(\mathbf{x}'',t'')\rangle,\end{aligned} \tag{25}$$

shown as a Feynman diagram in Fig. 1(b).

We note at this stage that the above procedure, i.e., integrating out the fermionic degrees of freedom after expanding the Hubbard-Stratonovich fields using second-order perturbation theory, recovers the random-phase approximation (RPA). The usual structure of the RPA response function contains the Stoner enhancement factor $1/(\Pi-U)$, where Π is the zeroth-order "bubble" diagram. This form applies to gapped fields such as the density-density and longitudinal spin-density–spin-density response functions, whose fluctuations we have neglected. The transverse spin-density–spin-density response function does not have this Stoner enhancement factor.

Next, we split the magnetization into semiclassical and fluctuating parts according to

$$\delta n_a(\mathbf{x},t_{\pm}) = \delta\Omega_a(\mathbf{x},t) \pm \frac{\xi_a(\mathbf{x},t)}{2}, \tag{26}$$

where t_+ and t_- refer to the forward and backward branches of the Keldysh contour, respectively. This transformation results in the action

$$\begin{aligned}S_{\text{eff}}[\delta\mathbf{\Omega},\boldsymbol{\xi}] = &\int_{-\infty}^{t} dt'\int d\mathbf{x}\left\{-\frac{\Delta\rho_s}{2}\delta\Omega_a(\mathbf{x},t')\xi_a(\mathbf{x},t')\right\} + \int_{-\infty}^{t} dt'\int d\mathbf{x}\int_{-\infty}^{t} dt''\int d\mathbf{x}'\{\delta\Omega_a(\mathbf{x},t')[\Pi^{(-)}_{ab}(\mathbf{x}-\mathbf{x}';t'-t'') \\ &+ K^{(-)}_{ab}(\mathbf{x}-\mathbf{x}';t'-t'')]\xi_b(\mathbf{x}',t'')\} + \int_{-\infty}^{t} dt'\int d\mathbf{x}\int_{-\infty}^{t} dt''\int d\mathbf{x}'\{\xi_a(\mathbf{x},t')[\Pi^{(+)}_{ab}(\mathbf{x}-\mathbf{x}';t'-t'') \\ &+ K^{(+)}_{ab}(\mathbf{x}-\mathbf{x}';t'-t'')]\delta\Omega_b(\mathbf{x}',t'')\} + \int_{-\infty}^{t} dt'\int d\mathbf{x}\int_{-\infty}^{t} dt''\int d\mathbf{x}'\{2\xi_a(\mathbf{x},t')[\Pi^{K}_{ab}(\mathbf{x}-\mathbf{x}';t'-t'') \\ &+ K^{K}_{ab}(\mathbf{x}-\mathbf{x}';t'-t'')]\xi_b(\mathbf{x}',t'')\},\end{aligned} \tag{27}$$

where the time integrations are now over the real axis from $-\infty$ to t.

Before we proceed, we make some general statements about dealing with functions on the Keldysh contour.[51] A general function $A(t,t')$, with time arguments on the Keldysh contour, can be decomposed into its analytic pieces by means of

$$A(t,t') \equiv \theta(t,t')A^{>}(t,t') + \theta(t',t)A^{<}(t,t'), \tag{28}$$

with $\theta(t,t')$ the Heaviside step function on the Keldysh contour. Generally, there can also be a piece $A^{\delta}\delta(t,t')$, but such a general decomposition is not needed here. Retarded and advanced functions, distinguished by the superscripts $(+)$ and $(-)$, respectively, are related to the analytic pieces by

$$A^{(\pm)}(t,t') \equiv \pm\,\theta(\pm(t-t'))[A^{>}(t,t') - A^{<}(t,t')]. \tag{29}$$

In addition, the Keldysh part, which, as we shall see, determines the strength of the fluctuations, is defined by

$$A^{K}(t,t') \equiv [A^{>}(t,t') + A^{<}(t,t')]. \tag{30}$$

Note that in the effective action [Eq. (27)], the retarded, advanced, and Keldysh parts of the various functions depend only on the difference of time arguments (we have implicitly taken the limit $\omega_p \to 0$).

To derive the equation of motion that is obeyed by the magnetization $\delta\Omega_a$, we perform another Hubbard-Stratonovich transformation and write the part of the action that is quadratic in the fluctuations ξ_a as a Gaussian functional integral over an auxiliary field η_a which will turn out to correspond, up to prefactors, to the stochastic magnetic field $\mathbf{h}$. Explicitly, we then have for the probability distribution that

$$\begin{aligned}P[\boldsymbol{\phi}^*,\boldsymbol{\phi},\hat{\Omega};t] \\ = \int d[\delta\mathbf{\Omega}]d[\boldsymbol{\xi}]d[\boldsymbol{\eta}]P[\boldsymbol{\eta}]\exp\left\{\frac{i}{\hbar}S_{\text{eff}}[\delta\mathbf{\Omega},\boldsymbol{\xi},\boldsymbol{\eta}]\right\},\end{aligned} \tag{31}$$

with the effective action

$$\begin{aligned}S_{\text{eff}}[\delta\mathbf{\Omega},\boldsymbol{\xi},\boldsymbol{\eta}] = &\int_{-\infty}^{t} dt'\int d\mathbf{x}\left\{-\frac{\Delta\rho_s}{2}\delta\Omega_a(\mathbf{x},t')\xi_a(\mathbf{x},t') + \eta_a(\mathbf{x},t)\xi_a(\mathbf{x},t)\right\} \\ &+ \int_{-\infty}^{t} dt'\int d\mathbf{x}\int_{-\infty}^{t} dt''\int d\mathbf{x}'\{\delta\Omega_a(\mathbf{x},t')[\Pi^{(-)}_{ab}(\mathbf{x}-\mathbf{x}';t'-t'') + K^{(-)}_{ab}(\mathbf{x}-\mathbf{x}';t'-t'')]\xi_b(\mathbf{x}',t'')\} \\ &+ \int_{-\infty}^{t} dt'\int d\mathbf{x}\int_{-\infty}^{t} dt''\int d\mathbf{x}'\{\xi_a(\mathbf{x},t')[\Pi^{(+)}_{ab}(\mathbf{x}-\mathbf{x}';t'-t'') + K^{(+)}_{ab}(\mathbf{x}-\mathbf{x}';t'-t'')]\delta\Omega_b(\mathbf{x}',t'')\}.\end{aligned} \tag{32}$$

This action is now linear in the fluctuations $\boldsymbol{\xi}$, and the functional integration over these fluctuations leads to a constraint that is precisely the equation of motion for the magnetization $\delta\boldsymbol{\Omega}$. We find that

$$-\left[\Pi_{ab}^{(+)}\left(-i\nabla,i\frac{\partial}{\partial t}\right)+\Pi_{ba}^{(-)}\left(i\nabla,-i\frac{\partial}{\partial t}\right)+K_{ab}^{(+)}\left(-i\nabla,i\frac{\partial}{\partial t}\right)+K_{ba}^{(-)}\left(i\nabla,-i\frac{\partial}{\partial t}\right)\right]\delta\Omega_b(\mathbf{x},t)+\frac{\Delta\rho_s}{2}\delta\Omega_a(\mathbf{x},t)=\eta_a(\mathbf{x},t), \tag{33}$$

with $\Pi_{ab}(\mathbf{q},\omega)$ and $K_{ab}(\mathbf{q},\omega)$ denoting the Fourier transforms of $\Pi_{ab}(\mathbf{x}-\mathbf{x}';t-t')$ and $K_{ab}(\mathbf{x}-\mathbf{x}';t-t')$, respectively. Note that the procedure used in Eqs. (26) and (31), which leads ultimately to the above equation of motion, circumvents the usual difficulties of deriving an equation of motion with dissipative terms from an action. The probability distribution for the noise is given by

$$P[\boldsymbol{\eta}]=\exp\left(\frac{i}{\hbar}\int_{-\infty}^{t}dt'\int d\mathbf{x}\int_{-\infty}^{t}dt''\int d\mathbf{x}'\{2\eta_a(\mathbf{x},t')\times[\Pi_{ab}^{K}(\mathbf{x}-\mathbf{x}';t'-t'')+K_{ab}^{K}(\mathbf{x}-\mathbf{x}';t'-t'')]^{-1}\eta_b(\mathbf{x}',t'')\}\right), \tag{34}$$

so that the correlation function of the stochastic magnetic field follows as

$$\langle\eta_a(\mathbf{x},t)\eta_b(\mathbf{x}',t')\rangle_{\text{noise}}=\frac{\hbar}{i}[\Pi_{ab}^{K}(\mathbf{x}-\mathbf{x}';t-t')+K_{ab}^{K}(\mathbf{x}-\mathbf{x}';t-t')]. \tag{35}$$

The results in Eqs. (33) and (35) are the main results of this section. Clearly, our main tasks are now to determine the long-wavelength low-frequency behavior of the nonequilibrium spin-density–spin-density response function and the photon–two-magnon vertex function. These calculations will be outlined in the next two sections. We start out with the equilibrium situation in which the electric field is zero and we only need to consider the spin-density–spin-density response function.

B. Equilibrium situation

As noted by Kohno *et al.*,[18] because the lifetime of the electrons is almost always extremely small compared to the spin splitting in metallic ferromagnets, i.e., $\hbar\gamma_\sigma \ll M$, we only need to consider the first vertex correction to the spin-density–spin-density response function. The diagrams that contribute to leading order are given in Fig. 2. The corresponding expression reads

$$\Pi_{ab}(\mathbf{q};t,t')=\frac{i\Delta^2}{8\hbar}\Bigg\{\int\frac{d\mathbf{p}}{(2\pi)^3}\mathrm{Tr}[\tau_a G(\mathbf{p}+\mathbf{q};t,t')\tau_b G(\mathbf{p};t',t)]+\frac{1}{\hbar^2}\sum_{a'\in\{0,x,y,z\}}\sigma_{a'}\int\frac{d\mathbf{p}_1}{(2\pi)^3}\int\frac{d\mathbf{p}_2}{(2\pi)^3}\int_{\mathcal{C}^\infty}dt''\times\int_{\mathcal{C}^\infty}dt'''\,\mathrm{Tr}[\tau_a G(\mathbf{p}_1+\mathbf{q};t,t'')\tau_{a'}G(\mathbf{p}_2+\mathbf{q};t'',t')\times\tau_b G(\mathbf{p}_2;t',t''')\tau_{a'}G(\mathbf{p}_1;t''',t)]\Bigg\}, \tag{36}$$

where the trace is over spin space and τ_0 denotes the 2×2 identity matrix. The first term in this equation corresponds to the lowest-order diagram in Fig. 2(a) and the second term to the diagram with the vertex correction in Fig. 2(b).

The Green's function is defined as

$$iG_{\sigma\sigma'}(\mathbf{x}-\mathbf{x}';t,t')\equiv\langle\psi_\sigma(\mathbf{x},t)\psi^*_{\sigma'}(\mathbf{x}',t')\rangle=\theta(t,t')\mathrm{Tr}[\hat{\rho}(-\infty)\hat{\psi}_\sigma(\mathbf{x},t)\hat{\psi}^\dagger_{\sigma'}(\mathbf{x}',t')]-\theta(t',t)\mathrm{Tr}[\hat{\rho}(-\infty)\hat{\psi}^\dagger_{\sigma'}(\mathbf{x}',t')\hat{\psi}_\sigma(\mathbf{x},t)], \tag{37}$$

so that the Fourier transforms of its analytic pieces read

$$-iG^{<}(\mathbf{k},\omega)=A(\mathbf{k},\omega)N_F(\hbar\omega-\mu),$$

$$iG^{>}(\mathbf{k},\omega)=A(\mathbf{k},\omega)[1-N_F(\hbar\omega-\mu)], \tag{38}$$

where the spectral function $A(\mathbf{k},\omega)$ is defined by

$$A(\mathbf{k},\omega)=i[G^{(+)}(\mathbf{k},\omega)-G^{(-)}(\mathbf{k},\omega)]. \tag{39}$$

Finally, the retarded and advanced Green's functions are given by

$$G^{(\pm)}_{\sigma\sigma'}(\mathbf{k},\omega)=\frac{\delta_{\sigma\sigma'}}{\hbar\omega^{\pm}-\epsilon_{\mathbf{k}}+M\sigma\pm i\hbar\gamma_\sigma}, \tag{40}$$

where $\hbar\omega^{\pm}=\hbar\omega\pm i0$ as usual.

With these ingredients, the calculation of the retarded, advanced, and Keldysh components of the response function in Eq. (36) is, in principle, straightforward. Some details of these calculations are described in the Appendix. Here, we directly present the results. For the retarded and advanced components, we find that

$$\Pi^{(\pm)}_{xx}(\mathbf{q},\omega)=\Pi^{(\pm)}_{yy}(\mathbf{q},\omega)=\frac{\Delta^2\rho_s}{8M}\pm i\pi\Delta^2\hbar\omega\left\{\frac{n_s u_s^2[\overline{S^2_\perp}(\nu_\uparrow^2+\nu_\downarrow^2)+2\overline{S^2_z}\nu_\uparrow\nu_\downarrow]}{8M^2}\right\},$$

$$\Pi^{(\pm)}_{xy}(\mathbf{q},\omega)=-\Pi^{(\pm)}_{yx}(\mathbf{q},\omega)=\frac{\Delta^2\rho_s}{16M^2}i\hbar\omega, \tag{41}$$

with the Keldysh parts given by

$$\Pi^K_{xx}(\mathbf{q},\omega) = \Pi^K_{yy}(\mathbf{q},\omega) = i\pi\Delta^2 k_B T\left\{\frac{n_s u_s^2[\overline{S_\perp^2}(\nu_\uparrow^2+\nu_\downarrow^2)+2\overline{S_z^2}\nu_\uparrow\nu_\downarrow]}{2M^2}\right\},$$

$$\Pi^K_{xy}(\mathbf{q},\omega) = \Pi^K_{yx}(\mathbf{q},\omega) = 0. \tag{42}$$

In order to obtain the Gilbert damping coefficient, it is sufficient to perform a zero-momentum small-frequency expansion of this response function. The first term that enters in a long-wavelength expansion is quadratic and determines the spin stiffness that is not of interest to use here. In addition, in order to determine the fluctuations, it turns out to be sufficient to obtain the zero-momentum zero-frequency part of the Keldysh response function. Inserting these results into the full equations of motion in Eqs. (33) and (35) straightforwardly leads to the results in Eqs. (10)–(14), with $\mathbf{v}_s=0$. (In arriving at these final results, we have taken the limit $\Delta_{\rm ext}\ll\Delta$.) In the next section, we consider the situation with an external electric field which leads to a nonzero spin-transfer velocity $\mathbf{v}_s$. We end this section by noting that, from a phenomenological viewpoint, Eqs. (10) and (14) are under debate,[36,38] even for $\mathbf{v}_s=0$. We hope that the microscopic derivation presented here sheds light on this controversy. Finally, we note that the temporal delta function in Eq. (14) arises by taking the zero-frequency limit of the Keldysh part of the spin-density–spin-density response function. This implies that the stochastic magnetic field in Eq. (14) corresponds to a Stratonovich stochastic process, rather an Ito one.[52]

C. With current

Our next task is to evaluate the function $K_{ab}(\mathbf{x};t,t')$ that is proportional to the photon–two-magnon interaction vertex and hence, from a microscopic point of view, ultimately gives rise to spin-transfer torques. The relevant Feynman diagrams are given in Fig. 3 and correspond to the expression

$$K_{ab}(\mathbf{q};t,t') = \frac{|e|\Delta^2}{4m\hbar}\int_{C^\infty}dt''\int\frac{d\mathbf{p}_1}{(2\pi)^3}\frac{e^{-i\omega_p t''}}{\omega_p}\Bigg(\mathrm{Tr}[\tau_a G(\mathbf{p}_1+\mathbf{q};t,t')\tau_b G(\mathbf{p}_1;t',t'')G(\mathbf{p}_1;t'',t)](\mathbf{p}_1\cdot\mathbf{E})$$
$$+\frac{1}{\hbar^2}\sum_{a'\in\{0,x,y,z\}}\sigma_{a'}\int_{C^\infty}dt'''\int_{C^\infty}dt''''\int\frac{d\mathbf{p}_2}{(2\pi)^3}\{\mathrm{Tr}[\tau_a G(\mathbf{p}_1+\mathbf{q};t,t'''')\tau_{a'}$$
$$\times G(\mathbf{p}_2+\mathbf{q};t'''',t')\tau_b G(\mathbf{p}_2;t',t''')\tau_{a'}G(\mathbf{p}_1;t''',t'')G(\mathbf{p}_1;t'',t)(\mathbf{p}_1\cdot\mathbf{E}) + \tau_a G(\mathbf{p}_1+\mathbf{q};t,t'''')\tau_{a'}$$
$$\times G(\mathbf{p}_2+\mathbf{q};t'''',t')\tau_b G(\mathbf{p}_2;t',t'')\tau_{a'}G(\mathbf{p}_2;t'',t''')G(\mathbf{p}_1;t''',t)(\mathbf{p}_2\cdot\mathbf{E})]\}\Bigg), \tag{43}$$

where the trace is again over spin space. In this expression, the first, second, and third terms correspond to the Feynman diagrams in Figs. 3(a)–3(c), respectively. Determining the low-frequency long-wavelength behavior of the retarded, advanced, and Keldysh components from Eq. (43) is straightforward but rather tedious. Typical steps in the calculations are illustrated in the Appendix for the spin-density–spin-density response function. Here, we directly present the results. Note that to obtain the spin-transfer torques and, in particular, the β coefficient that characterizes the nonadiabatic spin-transfer torque, it is sufficient to perform a zero-frequency long-wavelength expansion.

The results for the various parts of the function $K_{ab}(\mathbf{q};t,t')$, which ultimately determine the adiabatic spin-transfer torque, are given by

$$K^K_{xy}(\mathbf{q},\omega) = K^K_{yx}(\mathbf{q},\omega) = 0,$$

$$K^{(\pm)}_{xy}(\mathbf{q},\omega) = -K^{(\pm)}_{yx}(\mathbf{q},\omega) = \frac{i\Delta^2\hbar|e|(\mathbf{q}\cdot\mathbf{E})}{16mM^2}(n_\uparrow\tau_\uparrow^{\rm sc}-n_\downarrow\tau_\downarrow^{\rm sc}). \tag{44}$$

We note that these off-diagonal parts correspond to the reactive part of the photon–two-magnon interaction vertex. The dissipative part that gets contributions from decay processes and determines the nonadiabatic spin torque is given by

$$K^{(\pm)}_{xx}(\mathbf{q},\omega) = K^{(\pm)}_{yy}(\mathbf{q},\omega) = \pm\frac{i\Delta^2\hbar^2|e|(\mathbf{q}\cdot\mathbf{E})}{16mM^3}\Bigg[(n_\uparrow\tau_\uparrow^{\rm sc}\gamma_\downarrow - n_\downarrow\tau_\downarrow^{\rm sc}\gamma_\uparrow) - \frac{\pi}{\hbar}(n_i u^2 - n_s u_s^2\overline{S_z^2})(n_\uparrow\tau_\uparrow^{\rm sc}\nu_\downarrow - n_\downarrow\tau_\downarrow^{\rm sc}\nu_\uparrow)\Bigg],$$

$$K^K_{xx}(\mathbf{q},\omega) = K^K_{yy}(\mathbf{q},\omega) = 0. \tag{45}$$

From the above expression, we note that, to first order in the electric field, the Keldysh part of the photon–two-magnoninteraction vertex is zero. Ultimately, this implies that the current does not alter the thermal fluctuations, at least to first-order perturbation theory in the electric field. Finally, we remark that inserting the above results from Eqs. (44) and (45) into the general equation of motion in Eq. (33)

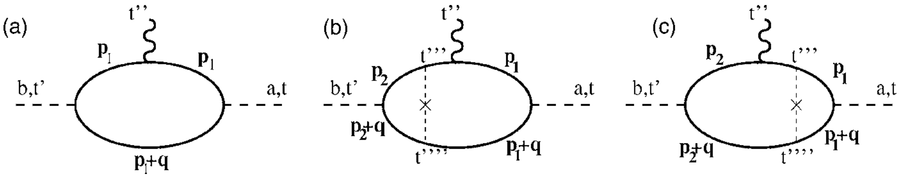

FIG. 3. Feynman diagrams that contribute to the spin-wave current interaction that gives rise to spin-transfer torques. (a) Lowest-order diagram, and [(b) and (c)] leading-order vertex corrections.

leads in a straightforward manner to the results for $\mathbf{v}_s$ and β presented in Sec. II.

IV. CONCLUSIONS

In conclusion, we have presented a general framework for the derivation of the effective equations of motion for the magnetization direction of a metallic ferromagnet, including nonzero-temperature effects and current. An important aspect of our approach is that the functional Keldysh methods we employ enable us to incorporate thermal fluctuations via stochastic forces in a unifying manner, without explicitly invoking the fluctuation-dissipation theorem. As a specific example, we have carried out detailed calculations for the model of a disordered itinerant ferromagnet used by Kohno *et al.*[18] Our results for the Gilbert damping parameter and the β parameter that characterizes the nonadiabatic torque are identical to the results found by these authors using imaginary-time methods. We have, in addition, determined the thermal fluctuations and found that, although the nonadiabatic torque corresponds to a dissipative process and the current-carrying situation makes application of the fluctuation-dissipation theorem questionable, to first order in the electric field, the usual fluctuation-dissipation relation to the Gilbert damping holds.

The method presented here is quite general, and in the near future, we intend to apply it also to other models of ferromagnets. We briefly comment on the generality of our results and what to expect for other models. In practice, ferromagnetism in metals is usually described in terms of some combination of ground-state and time-dependent spin-density-functional (SDF) theory. The structure of the ground-state theory is then the same as that of the saddle-point mean-field equations that arise in our theory, with the spin-dependent interaction in our theory replaced by the spin dependence of the exchange-correlation potential in SDF theory and our parabolic bands replaced by more complex bands specific to a particular material. For transition metals, the more realistic bands of SDF theory are hybridized s and d bands with $\mathbf{k}$ dependent spin splitting which tends to be larger in bands with dominant d character. Transition-metal ferromagnets are sometimes described by a crude model in which hybridization is not explicitly accounted for and the d orbitals are assumed to be fully spin polarized. In this s-d model, the d orbitals do not contribute to the density of states at the Fermi level since the majority spins are fully occupied and the minority spins are empty. It follows that the d orbitals do not contribute to transport or to any other property that involves only orbitals at the Fermi energy. When the formalism of our paper is applied to an s-d model rather than to the single-band model we discuss, the d orbitals can contribute to properties associated with the reactive pieces of the response functions we evaluate, but not to the properties that come from the low-energy limits of the dissipative response function pieces. The d orbitals do contribute to the coefficient of ω which translates into the LLG precessional dynamics time derivative, for example. [See the last line of Eq. (41). This $\mathbf{q}=\mathbf{0}$ time derivative can be interpreted as capturing the Berry phase associated with adiabatic spin dynamics.[53]] It follows that the d- and s-orbital contributions to this coefficient are proportional to their respective contributions to the total spin density. The d orbitals of an s-d model do *not*, however, contribute to the reactive adiabatic spin torque ($\mathbf{v}_s$) term because the d bands are either full or empty and therefore do not respond to an electric field.

The α and β dissipative parameters are both defined as dimensionless ratios of coefficient contributions from dissipative and nondissipative terms in the equation of motion [Eq. (41)]. Because α parametrizes the ratio of the two time-derivative terms, it is indirectly altered by the d bands. In contrast, β parametrizes the ratio of the two space-derivative terms neither of which has a d-orbital contribution. This is the reason, as noted in previous studies,[10,17,18] why β tends to be larger than α in s-d models, especially when the d orbitals make the dominant contribution to the spin density. As we have mentioned previously, and originally shown by Tserkovnyak *et al.*,[17] for spin-dependent scattering models with parabolic dispersion, $\alpha \approx \beta$ in a Stoner band model when the exchange splitting is much smaller than the Fermi energy but not when the d-orbital Berry phase is added to the reactive time derivative of an s-d model. It is perhaps expected that α should approximately equal to β in this limit since all the ingredients necessary for macroscopic Galilean invariance seem to be present. When the spin polarization is small, there is little to distinguish one direction of spin polarization from another and therefore one position in a spin texture from another. When the bands are parabolic in addition, an external electric field simply accelerates the system's center of mass. Explicit calculations for the present toy model demonstrate conclusively that the $\alpha=\beta$ condition which corresponds to macroscopic Galilean invariance is not generally satisfied. For more general spin-dependent disorder models or more realistic exchange splitting values, α and β are never equal.

These considerations do not directly apply to transition-metal ferromagnets because of s-d hybridization and because of the large d-orbital contribution to the minority-spin density of states. It is, nevertheless, true that the two reactive term coefficients can be expressed approximately as the sum of s- and d-orbital contributions. In the absence of spin-orbit coupling, the coefficient of $\partial\hat{\Omega}/\partial t$ in Eq. (1) is rigorously equal to 1 because the Berry phase is proportional to the total spin density, including the s and the dominant d contribution.

Similarly, the reactive coefficient of $\nabla\hat{\Omega}$ can be understood[54] in terms of the cancellation between convective and precessional contributions to spin dynamics in the static limit. It follows that the *d*-orbital weight in this reactive coefficient is not zero, as in the *s*-*d* model, but still relatively smaller than the *d* contribution to the reactive time-derivative coefficient. These considerations suggest that β/α will tend to be larger than 1 in most transition-metal ferromagnets. The main challenges in addressing this issue more quantitatively for a specific material are achieving an understanding of the nature of its spin-independent and spin-dependent disorders, accounting for the spin-orbit coupling present in the bands of the perfect crystal, and evaluating the vertex corrections (whose essential role is established by these toy model calculations) in systems with complex band structures.

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APPENDIX: CALCULATION OF THE RETARDED, IMAGINARY, AND KELDYSH PARTS OF THE SPIN-DENSITY–SPIN-DENSITY RESPONSE FUNCTION

The spin-density–spin-density response function in Eq. (36) is given by

$$\Pi_{ab}(\mathbf{q};t,t') = \Pi^0_{ab}(\mathbf{q};t,t') + \Pi^1_{ab}(\mathbf{q};t,t'), \tag{A1}$$

where the first term on the right-hand side is the lowest-order diagram in Fig. 2(a) and the second term is the vertex correction in Fig. 2(b). In the first part of this appendix, we evaluate the lowest-order diagram. In the second part, the vertex correction is calculated.

1. No vertex corrections

Without vertex corrections, the response function

$$\Pi^0_{ab}(\mathbf{q};t,t') = \frac{i\Delta^2}{8\hbar}\int\frac{d\mathbf{p}}{(2\pi)^3}\mathrm{Tr}[\tau_a G(\mathbf{p}+\mathbf{q};t,t')\tau_b G(\mathbf{p};t',t)], \tag{A2}$$

is shown diagrammatically in Fig. 2(a). The analytic pieces are now easily determined from the above result and given by

$$\Pi^{0,>}_{ab}(\mathbf{q};t,t') = \frac{i\Delta^2}{8\hbar}\int\frac{d\mathbf{p}}{(2\pi)^3} \times\mathrm{Tr}[\tau_a G^>(\mathbf{p}+\mathbf{q};t,t')\tau_b G^<(\mathbf{p};t',t)],$$

$$\Pi^{0,<}_{ab}(\mathbf{q};t,t') = \frac{i\Delta^2}{8\hbar}\int\frac{d\mathbf{p}}{(2\pi)^3} \times\mathrm{Tr}[\tau_a G^<(\mathbf{p}+\mathbf{q};t,t')\tau_b G^>(\mathbf{p};t',t)]. \tag{A3}$$

Using the results in Eqs. (37)–(40), we have for the retarded and advanced components of the zero-momentum Fourier-transformed response function that

$$\Pi^{0,(\pm)}_{ab}(\mathbf{q},\omega) = -\frac{\Delta^2}{8}\int\frac{d\epsilon}{(2\pi)}\int\frac{d\epsilon'}{(2\pi)}\int\frac{d\mathbf{p}}{(2\pi)^3} \times\left[\frac{N_F(\epsilon-\mu)-N_F(\epsilon'-\mu)}{\epsilon-\epsilon'-\hbar\omega^\pm}\right] \times\mathrm{Tr}[\tau_a A(\mathbf{p},\epsilon)\tau_b A(\mathbf{p},\epsilon')]. \tag{A4}$$

Expanding for small energies, we find that

$$\Pi^{0,(\pm)}_{ab}(\mathbf{q},\omega) \simeq -\frac{\Delta^2}{8}\int\frac{d\epsilon}{(2\pi)}\int\frac{d\epsilon'}{(2\pi)}\int\frac{d\mathbf{p}}{(2\pi)^3}[N_F(\epsilon-\mu)-N_F(\epsilon'-\mu)]\frac{\mathcal{P}}{(\epsilon-\epsilon')} \times\mathrm{Tr}[\tau_a A(\mathbf{p},\epsilon)\tau_b A(\mathbf{p},\epsilon')] - \frac{\Delta^2\hbar\omega}{8}\int\frac{d\epsilon}{(2\pi)}\int\frac{d\epsilon'}{(2\pi)}\int\frac{d\mathbf{p}}{(2\pi)^3}[N_F(\epsilon-\mu)-N_F(\epsilon'-\mu)]\frac{\mathcal{P}}{(\epsilon-\epsilon')^2} \times\mathrm{Tr}[\tau_a A(\mathbf{p},\epsilon)\tau_b A(\mathbf{p},\epsilon')] \mp \frac{i\Delta^2\hbar\omega}{32\pi}\int\frac{d\mathbf{p}}{(2\pi)^3}\mathrm{Tr}[\tau_a A(\mathbf{p},\mu)\tau_b A(\mathbf{p},\mu)]. \tag{A5}$$

From this, find $\Pi^{0,(\pm)}_{xx}(\mathbf{q},\omega)=\Pi^{0,(\pm)}_{yy}(\mathbf{q},\omega)$ and $\Pi^{0,(\pm)}_{xy}(\mathbf{q},\omega)=-\Pi^{0,(\pm)}_{yx}(\mathbf{q},\omega)$. Carrying out the remaining integrations, we have, in the limit $\gamma_\sigma/M\to 0$, that

$$\Pi^{0,(\pm)}_{xx}(\mathbf{q},\omega) = \frac{\Delta^2\rho_s}{8M} \pm \frac{i\Delta^2\pi\hbar\omega}{8}\left\{\frac{n_i u^2\nu_\uparrow\nu_\downarrow + n_s u_s^2[\overline{S_\perp^2}(\nu_\uparrow^2+\nu_\downarrow^2)+\overline{S_z^2}\nu_\uparrow\nu_\downarrow]}{M^2}\right\}, \quad \Pi^{0,(\pm)}_{xy}(0,\omega) = \frac{\Delta^2\rho_s}{16M^2}i\hbar\omega. \tag{A6}$$

The Keldysh component of the response function is, in first instance, given by

$$\Pi_{ab}^{0,K}(\mathbf{q},\omega) = \frac{\pi i \Delta^2}{4}\int \frac{d\epsilon}{(2\pi)}\int \frac{d\epsilon'}{(2\pi)}\int \frac{d\mathbf{p}}{(2\pi)^3}\delta(\hbar\omega-\epsilon+\epsilon')\times\{[1-N_F(\epsilon-\mu)]N_F(\epsilon'-\mu)+N_F(\epsilon-\mu)\times[1-N_F(\epsilon'-\mu)]\}\mathrm{Tr}[\tau_a A(\mathbf{p},\epsilon)\tau_b A(\mathbf{p},\epsilon')]. \tag{A7}$$

From this, we see that $\Pi_{xx}^{0,K}(\mathbf{q},\omega)=\Pi_{yy}^{0,K}(\mathbf{q},\omega)$ and $\Pi_{xy}^{0,K}(\mathbf{q},\omega)=\Pi_{yx}^{0,K}(\mathbf{q},\omega)=0$. We find that

$$\Pi_{xx}^{0,K}(\mathbf{q},\omega) = -\frac{i\pi\Delta^2 k_B T}{2}\left\{\frac{n_i u^2\nu_\uparrow\nu_\downarrow + n_s u_s^2[\overline{S_\perp^2}(\nu_\uparrow^2+\nu_\downarrow^2)+\overline{S_z^2}\nu_\uparrow\nu_\downarrow]}{M^2}\right\}. \tag{A8}$$

Moreover, we have that

$$\Pi_{xx}^{0,K}(\mathbf{q},\omega) = \pm 2i[2N_B(\hbar\omega)+1]\mathrm{Im}\,\Pi_{xx}^{0,(\pm)}(\mathbf{q},\omega), \tag{A9}$$

where $N_B(x)$ is the Bose distribution function. This is the fluctuation-dissipation theorem, which emerges naturally from the formalism.

2. Vertex correction

The first-order vertex correction is shown in Fig. 2(b) and given by

$$\Pi_{ab}^{1}(\mathbf{q};t,t') = \frac{i\Delta^2}{8\hbar^3}\sum_{a'\in\{0,x,y,z\}}\sigma_{a'}\int\frac{d\mathbf{p}_1}{(2\pi)^3}\int\frac{d\mathbf{p}_2}{(2\pi)^3}\int_{C^\infty}dt''\int_{C^\infty}dt'''\times\mathrm{Tr}[\tau_a G(\mathbf{q}+\mathbf{p}_1;t,t''')\tau_{a'}G(\mathbf{q}+\mathbf{p}_2;t''',t')\tau_b\times G(\mathbf{p}_2;t',t'')\tau_{a'}G(\mathbf{p}_1;t'',t)]. \tag{A10}$$

Before we proceed, we state some rules for calculus involving the Keldysh contour. From an equation like

$$A(t,t') = \int_{C^\infty} dt'' B(t,t'')C(t'',t'), \tag{A11}$$

we find the analytic pieces as

$$A^{\gtrless}(t,t') = \int_{-\infty}^{\infty} dt'' B^{(+)}(t,t'')C^{\gtrless}(t'',t') + \int_{-\infty}^{\infty} dt'' B^{\gtrless}(t,t'')C^{(-)}(t'',t'), \tag{A12}$$

where the retarded and advanced components are defined in Eq. (29). It is then also straightforward to show that

$$A^{(\pm)}(t,t') = \int_{-\infty}^{\infty} dt'' B^{(\pm)}(t,t'')C^{(\pm)}(t'',t'). \tag{A13}$$

Using these rules, we find from Eq. (A10)

$$\Pi_{ab}^{1,(\pm)}(\mathbf{q},\omega) = -\frac{\Delta^2}{8}\sum_{a'\in\{0,x,y,z\}}\sigma_{a'}\int\frac{d\epsilon}{(2\pi)}\int\frac{d\epsilon'}{(2\pi)}\int\frac{d\mathbf{p}_1}{(2\pi)^3}\int\frac{d\mathbf{p}_2}{(2\pi)^3}\frac{1}{\hbar\omega^\pm-\epsilon+\epsilon'}\mathrm{Tr}\{[\tau_a G^{(+)}(\mathbf{q}+\mathbf{p}_1;\epsilon)\tau_{a'}G^{>}(\mathbf{q}+\mathbf{p}_2;\epsilon) + \tau_a G^{>}(\mathbf{q}+\mathbf{p}_1;\epsilon)\tau_{a'}G^{(-)}(\mathbf{q}+\mathbf{p}_2;\epsilon)][\tau_b G^{(+)}(\mathbf{p}_2;\epsilon')\tau_{a'}G^{<}(\mathbf{p}_1;\epsilon') + \tau_b G^{<}(\mathbf{p}_2;\epsilon')\tau_{a'}G^{(-)}(\mathbf{p}_1;\epsilon')] - [\tau_a G^{(+)}(\mathbf{q}+\mathbf{p}_1;\epsilon)\tau_{a'}G^{<}(\mathbf{q}+\mathbf{p}_2;\epsilon) + \tau_a G^{<}(\mathbf{q}+\mathbf{p}_1;\epsilon)\tau_{a'}G^{(-)}(\mathbf{q}+\mathbf{p}_2;\epsilon)][\tau_b G^{(+)}(\mathbf{p}_2;\epsilon')\tau_{a'}G^{>}(\mathbf{p}_1;\epsilon') + \tau_b G^{>}(\mathbf{p}_2;\epsilon')\tau_{a'}G^{(-)}(\mathbf{p}_1;\epsilon')]\}. \tag{A14}$$

The Keldysh component reads

$$\Pi_{ab}^{1,K}(\mathbf{q},\omega) = \frac{i\pi\Delta^2}{4}\sum_{a'\in\{0,x,y,z\}}\sigma_{a'}\int\frac{d\epsilon}{(2\pi)}\int\frac{d\epsilon'}{(2\pi)}\int\frac{d\mathbf{p}_1}{(2\pi)^3}\int\frac{d\mathbf{p}_2}{(2\pi)^3}\delta(\hbar\omega-\epsilon+\epsilon')\mathrm{Tr}\{[\tau_a G^{(+)}(\mathbf{q}+\mathbf{p}_1;\epsilon)\tau_{a'}G^{>}(\mathbf{q}+\mathbf{p}_2;\epsilon) + \tau_a G^{>}(\mathbf{q}+\mathbf{p}_1;\epsilon)\tau_{a'}G^{(-)}(\mathbf{q}+\mathbf{p}_2;\epsilon)][\tau_b G^{(+)}(\mathbf{p}_2;\epsilon')\tau_{a'}G^{<}(\mathbf{p}_1;\epsilon') + \tau_b G^{<}(\mathbf{p}_2;\epsilon')\tau_{a'}G^{(-)}(\mathbf{p}_1;\epsilon')] + [\tau_a G^{(+)}(\mathbf{q}+\mathbf{p}_1;\epsilon)\tau_{a'}G^{<}(\mathbf{q}+\mathbf{p}_2;\epsilon) + \tau_a G^{<}(\mathbf{q}+\mathbf{p}_1;\epsilon)\tau_{a'}G^{(-)}(\mathbf{q}+\mathbf{p}_2;\epsilon)][\tau_b G^{(+)}(\mathbf{p}_2;\epsilon')\tau_{a'}G^{>}(\mathbf{p}_1;\epsilon') + \tau_b G^{>}(\mathbf{p}_2;\epsilon')\tau_{a'}G^{(-)}(\mathbf{p}_1;\epsilon')]\}. \tag{A15}$$

We focus now on the imaginary part of $\Pi_{xx}^{1,(\pm)}(\mathbf{q},\omega)=\Pi_{yy}^{1,(\pm)}(\mathbf{q},\omega)$ since this determines the Gilbert damping constant. Carrying out the momentum and energy integrals results in

$$\Pi_{xx}^{1,(\pm)}(\mathbf{q},\omega) = \mp\frac{i\pi\Delta^2\hbar\omega}{8}\left[\frac{n_i u^2\nu_\uparrow\nu_\downarrow - n_s u_s^2\overline{S_z^2}\nu_\uparrow\nu_\downarrow}{M^2}\right]. \tag{A16}$$

The corresponding Keldysh part is given by

$$\Pi^{1,K}_{xx}(\mathbf{q},\omega) = \frac{\pi i \Delta^2 k_B T}{2}\left[\frac{n_i u^2 \nu_\uparrow \nu_\downarrow - n_s u_s^2 \overline{S_z^2} \nu_\uparrow \nu_\downarrow}{M^2}\right]. \quad \text{(A17)}$$

Adding the results from Eqs. (A6) and (A16), we get the result presented in Eq. (41). Similarly, adding the result in Eq. (A8) to Eq. (A17) reproduces Eq. (42).

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